and y_2, y_3 we also have

$$m>n.$$

If in (6) we put $n=\mu-\frac{1}{2}$ we have,

$$\frac{d^2y}{dt^2}-\frac{2t}{1-t^2}\frac{dy}{dt}+\frac{[(\mu^2-\frac{1}{4})(1-t^2)-m^2]y}{(1-t^2)^2}=0\ldots.(7),$$

which is (Craig, page 194) the differential equation for Hicks' Toroidal functions.

We have for this case

$$y_1=\int_0^1(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{2}(2m+2\mu+1)}\,du,$$

$$y_2=\int_0^{-\infty}(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{2}(2m+2\mu+1)}\,du,$$

$$y_3=\int_1^{+\infty}(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{2}(2m+2\mu+1)}\,du,$$

where $\mu+\frac{1}{2}>0$, and in y_2, y_3 $m>\mu-\frac{1}{2}$.

FORECASTING THE CENSUS RETURNS.

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Now that the government of the United States is about to take another census, we occasionally see in the newspapers forecasts of the population. Sometimes these forecasts are mere guesses, but there is a method by which one can make these estimates scientifically, and if they fail to come out right it is the fault of the people rather than the method.

All physical laws may be divided into two classes—rational and empirical. The free fall of a body, the swinging of a pendulum, and most of the laws of heat and electricity illustrate the first class. If a body falls one space the first second it will fall three the second and five the third. These laws may easily be embodied into formulae which contain no arbitrary constants. On the other hand, such problems as the relation between temperature and depth below the

surface, and the gravitational acceleration over the earth's surface follow no fixed law. In order to express them at all, special formulae must be devised which contain constants whose values are to be determined.

It is obvious that the law of increase of population of a country belongs to this latter class. To construct an empirical formula a curve is plotted showing the relation between the various decades and their corresponding population. When this is drawn, unless it is too irregular, a mathematician can locate it among the curves with which he is familiar, and the equation of the curve is the empirical formula required. The census curve for the United States turns out to be a parabola whose equation is $p=S+Tx+Ux^2$ in which p is the population, x the number of the decade, and S, T, and U are unknown constants.

To determine these constants we first write down the population for the various decades. (These figures are from the World Almanac.)

1790	—	3.6
1800	—	5.3
1810	—	7.2
1820	—	9.6
1830	—	12.9
1840	—	17.1
1850	—	23.2
1860	—	31.4
1870	—	38.6
1880	—	50.2
1890	—	62.6

The population is expressed in millions and tenths of a million. Substituting these values in our formula we have:

$$\begin{aligned}
3.6&=S-T+U\\
5.3&=S+0+0\\
7.2&=S+T+U\\
9.6&=S+2T+4U\\
12.9&=S+3T+9U\\
17.1&=S+4T+16U\\
23.2&=S+5T+25U\\
31.4&=S+6T+36U\\
38.6&=S+7T+49U\\
50.2&=S+8T+64U\\
62.6&=S+9T+81U
\end{aligned}$$

These are called "observation equations." If we multiply each one in turn by the coefficients of S, T, and U respectively, and add the results, we have what are called "normal equations." They are as follows:

$$
\begin{aligned}
-3.6 &= -S + T - U \\
7.2 &= S + T + U \\
19.2 &= 2S + 4T + 8U \\
38.7 &= 3S + 9T + 27U \\
68.4 &= 4S + 16T + 64U \\
116.0 &= 5S + 25T + 125U \\
188.4 &= 6S + 36T + 216U \\
270.2 &= 7S + 49T + 343U \\
401.6 &= 8S + 64T + 512U \\
563.4 &= 9S + 81T + 728U \\
\hline
1669.5 &= 44S + 286T + 2023U
\end{aligned}
$$

$$
\begin{aligned}
3.6 &= S - T + U \\
7.2 &= S + T + U \\
38.4 &= 4S + 8T + 16U \\
116.1 &= 9S + 27T + 81U \\
273.6 &= 16S + 64T + 256U \\
580.0 &= 25S + 125T + 625U \\
1130.4 &= 36S + 216T + 1296U \\
1891.4 &= 49S + 343T + 2401U \\
3212.8 &= 64S + 512T + 4096U \\
5070.6 &= 81S + 729T + 6561U \\
\hline
12324.1 &= 286S + 2026T + 15334U
\end{aligned}
$$

When we multiply through by the coefficients of S (unity) the observation equations undergo no change and their sum is :

$$261.7 = 11S + 44T + 286U.$$

We have now three equations with three unknown quantities which we may solve in any manner we choose. The values obtained are :

$$
\begin{aligned}
S &= 6.08 \\
T &= 0.690 \\
U &= 0.622
\end{aligned}
$$

If we substitute these in the formula we get:

$p = 6.08 + 6.9 + 62.2$, or
$p = 75.2$ millions, which is the forecast for 1900.

If we neglect the curves of 1890 the formula becomes :

$$p = 4.97 + 0.873x + 0.581x^2.$$

If we use only those from 1820 to 1880 inclusive we get:

$$p=7.29-0.28x+0.689x^2.$$

For the year 1894 the first of the formula gave 64.5 and the second 65.5; while the population reported in the World Almanac was 66.7.

The second formula gives for 1900 a population of 73.4, while the first formula gives a result somewhat lower. It appears then, that the population is increasing more rapidly than the parabolic curve indicates, and that if anything our forecast of 73.4 is somewhat low. It must be understood of course that the above formulae are strictly anti-expansion and make no allowance for our new possessions.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

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[Continued from the February Number.]

CHAPTER VI.

GEOMETRICAL ADDITION AND SUBTRACTION.

73. The general theory of the *Ausdehnungslehre* may be applied in such diverse sciences as geometry, mechanics, and logic. We proceed in this and the following chapter to apply it in geometry.

74. The concepts dealt with in geometry are the *point*, *line*, *surface*, and *solid*, which may or may not be fixed in position. For the sake of distinction a line whose length and direction are fixed but not its position is called a *vector*. (3). A portion of a plane whose direction and extent are fixed but not the position of the plane is called by analogy a *plane vector*.

75. As an introduction to the *Ausdehnungslehre* the addition and subtraction of vectors was treated in Chapter I. It is evident from what was given in that chapter that plane vectors may be added and subtracted in the same way as line vectors. One gets the parts whose sum is a given plane vector by projecting the given plane vector on the coördinate planes.

As we have already treated of the addition and subtraction of vectors, we proceed to apply the laws of addition and subtraction to points.

76. We will define a point as an infinitesimal portion of a line and denote it by p. When the point has position we will denote it by $p\rho$, in which p denotes the point at the extremity of the radius vector ρ from the origin, O. Evidently a line, or plane, or solid may be located by means of a radius vector in the same way. (See 168.)